Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 28

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Textbook Sections: Review of Chapter 11

A Review of Sequences, Series, and Taylor Series

Part I: Sequences: (Lectures 15 – 16)

Def: A sequence is an ordered list of numbers (or, equivalently, a function from the positive integers $\mathbb{N}=\{1,2,3,\dots\} \to \mathbb{R}$).

$$\{a_n\}_{n=1}^{\infty} \quad \text{or} \quad a_1, a_2, a_3, \dots$$

Here $a_n \in \mathbb{R}$ is the number assigned to the positive integer $n \in \mathbb{N}$ (we call a_n the n^{th} term of the sequence).

Main Def: A sequence $\{a_n\}_1^{\infty}$ has limit $L \in \mathbb{R}$ provided:

For every choice of small number $\varepsilon > 0$ there is a big enough "time" $N \in \mathbb{N}$ so that for every $n \geq N$,

$$L - \varepsilon \leq a_n \leq L + \varepsilon.$$

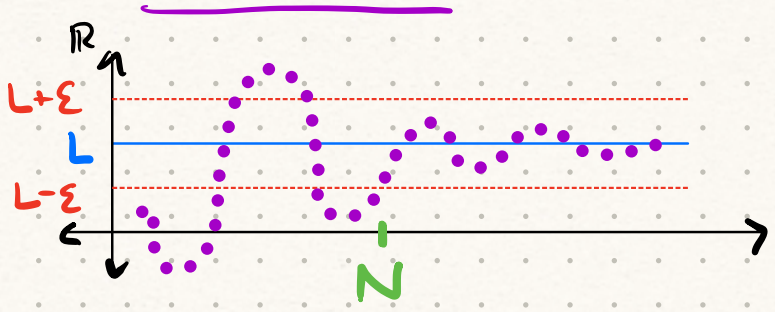
In this case, we write $\lim_{n\to\infty} a_n = L$.

What's this mean?

In words:

a_n is as close to L as we want, as long as we wait long enough (i.e. let n be big enough).

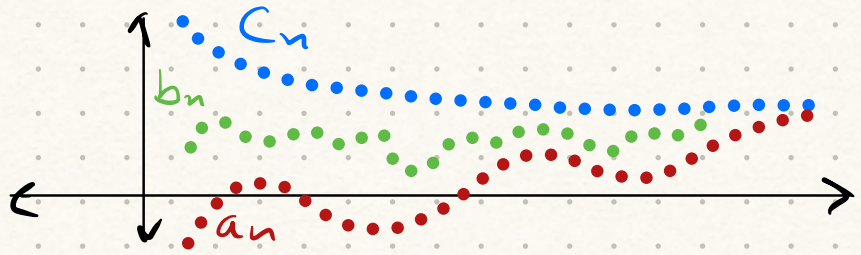
In Pictures



We introduced a number of "Limit Laws" which tell us rules for computing with limits of sequences. One of the key tools we introduced for finding limits was:

Squeeze Theorem Suppose $a_n \le b_n \le c_n$ for all $n \ge 1$.
If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ too.

In pictures:



We also defined monotonicity, and learned that bounded, monotone sequences are always convergent.

Part II: Series of Real Numbers (Lectures 17-21)

Definition: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. The "sum" of all the terms a_n is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \qquad \text{(or } \textstyle\sum a_n \text{ for short)}$$

and is called an infinite series.

Rigorously: Let $S_N = \sum_{n=1}^{N} a_n$ denote the N^{th} partial sum. Then

$$\sum_{n=1}^{\infty} a_n := \lim_{N\to\infty} S_N \quad \text{if it exists.}$$

Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

As a key example: we considered the geometric series

$$\sum_{n=0}^{\infty} a r^n.$$

It has $S_N = a\dfrac{1-r^{N-1}}{1-r}$, so that $\displaystyle\sum_{n=0}^{\infty} a r^n = \begin{cases} \lim_{N\to\infty} S_N = \frac{a}{1-r} & |r|<1 \\ \text{diverges} & |r|\ge 1 \end{cases}$.

To study the convergence/divergence of a given series $\sum_{n=1}^{\infty} a_n$, we built up a toolbox of tests to apply:

1) Does $\lim_{n\to\infty} a_n = 0$? If <u>not</u>, then $\sum a_n$ <u>diverges</u>.

2) Do the terms a_n alternate in sign? $\longmapsto$ Alternating Series Test

$\sum a_n = \sum (-1)^n b_n$ with $b_n \geq 0$, $b_n \geq b_{n+1}$, $b_n \to 0$ $\Rightarrow$ conv.

3) Can we estimate $a_n \leq b_n$, or $c_n \leq a_n$? $\longmapsto$ Comparison Laws

$\Rightarrow \sum c_n \leq \sum a_n \leq \sum b_n$ $\rightsquigarrow$ we can control $\sum a_n$ using $\sum b_n$ and $\sum c_n$.

4) Can we estimate $\frac{a_n}{b_n} \to c$? $\longmapsto$ Limit Comparison Test

$\sum a_n, \sum b_n$ with $a_n, b_n > 0$, $\frac{a_n}{b_n} \to c \in (0,\infty) \Rightarrow$ both conv. or both div.

5) Can we write $a_n = f(n)$ for $f(x)$ continuous? $\longmapsto$ Integral Test

$a_n = f(n)$ with $f : [1,\infty) \to \mathbb{R}$ continuous, eventually positive and decreasing

$\Rightarrow \sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

We can also use the integral test to estimate the N^{th} remainder $R_N = \sum a_n - S_N$.

6) Can we compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|}$? $\longmapsto$ Ratio/Root Test

For either test: if $\begin{cases} \lim(\cdots) < 1 \Rightarrow \sum a_n \text{ converges absolutely} \\ \lim(\cdots) > 1 \Rightarrow \sum a_n \text{ diverges} \\ \lim(\cdots) = 1 \Rightarrow \text{test is inconclusive} \end{cases}$

<u>Another key example</u>: p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Just like for sequences, we also have several "series laws" which tell us how to add, subtract, and scale infinite series.

Part III: Power Series (Lectures 22-24)

We broaden our point of view by considering infinite series made out of functions instead of just real numbers:

Def: A power series is a series of the form

$$\sum_{n=1}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where we think of x as a variable in some domain of convergence.

Theorem: Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Then exactly one of the following occurs:

1) The series converges for every $x \in \mathbb{R}$. In this case, we call $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ an analytic function.
2) The series only converges when $x=a$.
3) There exists a finite "radius of convergence" $R>0$ so that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges when $|x-a|<R$ and diverges when $|x-a|>R$. $f(x)$ can converge or diverge at $x = a \pm R$!

Key Tools: To find R (when given a power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$) we can try the ratio and root tests, computing

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a| \quad \text{or} \quad \lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = \lim_{n\to\infty} \sqrt[n]{|c_n|}\,|x-a|.$$

and then finding $R>0$ so that $|x-a|<R$ makes these limits <1. At the endpoints, we investigate using other tests.

As for calculus...

Inside the domain of convergence, we can differentiate and integrate a power series term-by-term, just like we do for polynomials:

Theorem: (Term-by-term Diff'n and Int'n)

Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$.

Then

(1) $f(x)$ is differentiable on $(a-R, a+R)$, where

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n (x-a)^n\right) = \sum_{n=0}^{\infty} c_n \cdot n (x-a)^{n-1} = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$$

This series representation of $f'(x)$ also has radius of convergence R.

(2) $f(x)$ has an antiderivative on $(a-R, a+R)$:

$$F(x) = \int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$$

This series representation of $F(x)$ also has radius of convergence R.

Part IV: Taylor Series (Lectures 25-27)

Theorem: If $f(x)$ has a power series representation on $|x-a| < R$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then we must have

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

We call

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

the Taylor Series of $f(x)$ at a.

Q: When does a function $f(x)$ equal its Taylor series?

If $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ for $|x-a| < R$, then by definition

$$f(x) = \lim_{N\to\infty} T_N(x) := \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n}_{N^{th} \text{ degree Taylor Polynomial}}.$$

define the remainder, which is now a function of x:

$$R_N(x) = f(x) - T_N(x) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Key Question: As $N \to \infty$, does $|R_N(x)| \to 0$ for all $|x-a| < R$?

This is what it takes to have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $|x-a| < R$.

Key Tool: To understand when $|R_N(x)| \to 0$, the following is often useful.

Theorem: (Taylor's Inequality) Suppose $f^{(n+1)}(x)$ is continuous.

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le R$, then

$$|R_N(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{on } |x-a| \le R$$

one can do multiplication and long division with Taylor series, just like for polynomials:

Theorem: Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$ are both convergent power series on $|x-a| < R$

① $f(x) \cdot g(x) = \left(\sum_0^\infty c_n(x-a)^n\right)\left(\sum_0^\infty d_n(x-a)^n\right)$

$= c_0d_0 + (c_1d_0 + c_0d_1)(x-a) + (c_2d_0 + c_1d_1 + c_0d_2)(x-a)^2 + \cdots$

converges on $|x-a| < R$.

② If $g(a) = d_0 \neq 0$, then

$$\frac{f(x)}{g(x)} = \frac{\sum_0^\infty c_n(x-a)^n}{\sum_0^\infty d_n(x-a)^n} = \text{--- long division ---}$$

converges for $|x-a| < r < R$.

Example Problems:

1) $\lim \frac{3n+1}{n}$?

2) $\lim \frac{\sin(n\pi)}{n^{1/2}}$?

3) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ sq. $\quad 0 \le x^2 \sin\left(\frac{1}{x}\right) \le x \cdot \frac{2}{x} = x$

4) $\lim \frac{\ln n}{n}$ comp $\quad \ln n \le n^{1/2}$ as $n \to \infty$

5) $\sum_{n=1}^{\infty} \frac{n+1}{n^{5/2}}$ p-test

6) $\sum_{n=-2}^{\infty} \left(\frac{2}{3}\right)^n$

7) $\sum_{1}^{\infty} \frac{n!}{3^n}$ ratio

8) $\sum_{1}^{\infty} \left(\frac{5}{n}\right)^n$ root

9) $\sum \frac{1}{n(\ln n)^2}$ int. $\quad \int_2^{\infty} \frac{1}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{1}{u^2} = \frac{1}{\ln 2}$

10) $\sum \frac{\sin(n\pi)}{n^{1/2}}$

11) $\sum a_n = 2, \quad \sum b_n = 5 \quad \rightsquigarrow \quad \sum (a_n - 5b_n) = ?$

12) radius of conv: $\sum \frac{x^n}{(n!)^2}$

13) " " $\sum \frac{(-1)^n x^n}{n}$ $\quad |x| < 1$

14) $f(x) = \sum_0^{\infty} x^{2n}$ on $|x| < 1$.
find $f'(x)$ and $\int_0^x f(t)\,dt$

15) Find the 2^{nd} order coeff. of the Taylor series of $e^{\sin x}$ at 0.

16) Find the 1^{st} three nonzero terms of $e^x \sin x$